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A survey on networking games in telecommunications

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Abstract

In this survey, we summarize different modeling and solution concepts of networking games, as well as a number of different applications in telecommunications that make use of or can make use of networking games. We identify some of the mathematical challenges and methodologies that are involved in these problems. We include here work that has relevance to networking games in telecommunications from other areas, in particular from transportation planning.

Keywords: Game theory; Telecommunication

1. Introduction

With the deregulation of the telecommunication companies and the rapid growth of the Internet, the research area of networking games has experienced a remarkable development. The impetus to this surge of research is the clear limitation in the telecom and internet industries of the pure optimization approach, with respect to routing, resource or quality of service allocation and pricing. Indeed, the optimization approach assumes that the goal of the routing strategy, allocation, or price choices can be defined independently of the reactions of other actors, users, or players, in the industry. At nearly all levels of the decision process, however, interaction across players is non-negligible, where *players* may refer to other telecom firms, internet service providers, or even users themselves, who vie for limited resources. When interactions are to be taken into account, because the choices of any one actor influence the choices of the others, a

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natural modeling framework involves seeking an equilibrium, or stable operating point, of the system. In this setting, each actor seeks to optimize her or his own criterion, which includes the influence of the decisions of the other actors upon his own, and all actors perform this optimization simultaneously. The Nash equilibrium concept is one example of this, which has been extended to networks. However, as we shall see in this survey, it is not the only such concept. In particular, many modeling aspects from the study of equilibrium in transportation networks have been successfully applied to telecommunications.

Equilibrium models in transportation networks have been studied for 50 years, since their introduction in 1952 [1], and many extensions and variations of this concept exist; most, though, have yet to be carried over to the telecommunication arena. We shall highlight some particularly promising extensions in this survey. At the same time, some very similar concepts appear to have emerged in game theory in the past 10 years; these too will be discussed here.

One clear need in the field of networking games in telecommunications is therefore to make the most of research results of these different communities: mathematics, economics, information sciences and transportation engineering. Another is to continue defining new problems and models from the point of view of telecommunications technology, problems that may not before have been posed for lack of an appropriate modeling paradigm, but that may lend themselves to the network equilibrium framework.

In preparing this survey on networking games in telecommunications, we attempted to summarize the different modeling and solution concepts, and to highlight the different types of applications in which networking games are useful in telecommunications, as well as to identify some of the mathematical challenges that are involved in these problems. With respect to telecommunication applications, we have encountered a rich literature in flow and congestion control [2–15,182–186], network routing [16–37,1], file allocation [116], load balancing [39–43], multi-commodity flow [44,45], resource allocation [46–50] and quality of service provisioning [50,51], see also [52–54]. Some papers have considered the combination of flow and routing in a non-cooperative setting; see [55–59,35,60] and references therein. As shown in [35] in a compendium of transportation equilibrium models and algorithms, when the objective functions of the players are the sum of link costs plus a reward which is a function of the throughput, then the underlying game can be transformed into one involving only routing decisions.

A promising potential application of game theory is the area of network security, see [61] and [62]. Intensive research effort has also been devoted to game models in wireless networks. Some of the main issues there are power control [63–71], pricing and incentive for cooperation between mobile terminals [72–75], security issues [62], the access control to a common shared radio channel [76–78], and auctions for resource reservation [79]. We shall not attempt to review the area of networking games in wireless networks in this survey. Some other related surveys are [80], as well as a whole special issue of the journal *Networks and Spatial Economics* on *Crossovers between Transportation Planning and Telecommunications*, to appear in 2003.

In this survey we focus primarily on non-cooperative games. We discuss different equilibrium concepts, in terms both of their qualitative and quantitative properties. In particular, we consider in depth the issue of uniqueness of an equilibrium, the Braess paradox, controlling equilibria through design parameters or pricing, as well as the Stackelberg framework for hierarchical, or leader–follower, equilibrium. We provide as well a brief summary of some work on equilibria in cooperative games that are related to resource allocation, pricing and to the Stackelberg framework.

The structure of the survey is as follows. We begin in Section 2 by presenting basic notions of game theory related to this survey. We present there the notions of multi-criteria and hierarchical equilibria as well as potential games. We then describe in Section 3 the state of the art in non-cooperative service provisioning and routing in networks. In Section 4 we discuss the work on non-cooperative flow control. In Section 5 we discuss the uniqueness of equilibrium and in Section 6 we describe issues related to convergence to equilibrium from initial non-equilibria strategies. Then we survey in Section 7 issues related to some properties of equilibria and the way they can be influenced by network architecting and administration, which includes the discussion of the Braess paradox, hierarchical games and pricing issues. We conclude with the topic of cooperative equilibria in telecommunications.

2. Basic game concepts

In this section we introduce the basic definitions and notation needed by the equilibrium models that have been studied in communication networks.

As the primary focus of the survey is the non-cooperative framework, in which each user optimizes her or his decision in an individual way, we begin by presenting the non-cooperative Nash equilibrium.

2.1. Nash equilibrium and its variants

Let us consider a model with n users, each of whom attempts to maximize his own particular utility function; denote the utility function of user i as J^i. Further, let u^i denote the decision, or *strategy*, of user i and u^{-i} the strategies of all users other than user i. The utility function of user i is expressed as a function both of the vector of strategies of all users, $\mathbf{u} = (u^1, \ldots, u^n)$, and of a vector of system, or control, parameters, x, that is, $J^i(\mathbf{u}, x)$.

For x fixed, we say that $\mathbf{u}^*(x) = (u^{1*}, \ldots, u^{n*})$ is a Nash equilibrium if no user can improve her or his utility by unilateral deviation. More precisely, for each $i \in \{1, 2, \ldots, n\}$, a Nash equilibrium satisfies

$$\begin{aligned} J^i(\mathbf{u}^*(x), x) &= \max_{u^i} J^i(u^{1*}, \ldots, u^{i-1*}, u^i, u^{i+1*}, \ldots, u^{n*}, x) \\ &= \max_{u^i} J^i(u^{-i*}, u^i, x). \end{aligned} \tag{1}$$

In practice, a user may have constraints on her or his strategy, and this gives rise to constrained Nash equilibria. One example is the so-called “coupled constraint” set of [81]. Denote

$$\Pi(x) = \{\mathbf{u} : g_l(\mathbf{u}, x) \geqslant 0, \ \ l = 1, \ldots, k\},$$

the set of n-tuple actions of the n users that satisfy the $k \times k'$ constraints, where $g_l(., x)$ is a mapping of $\mathbb{R}^n \to \mathbb{R}^{k'}$, with each component of g_l being a convex function. In the special case where the constraint sets are orthogonal we have $\Pi(x) = \Pi^1(x) \times \Pi^2(x) \times \cdots \times \Pi^n(x)$, where $\Pi^i(x) = \{\mathbf{u} : g_l^i(\mathbf{u}^i, x) \geqslant 0, \ \ l = 2, \ldots, k_i\}$ is the set of actions that satisfy the k_i constraints for user i. The number of orthogonal constraints imposed on each decision problem may vary across users, where that number is referred to as k_i. The vector $\mathbf{u}^*$ is then said to be a constrained Nash equilibrium if $\mathbf{u}^* \in \Pi(x)$, and, in addition,

$$J^i(\mathbf{u}^*(x), x) = \max_{u_i}(J^i(u^{-i*}, u^i, x) \text{ such that } (u^{-i*}, u^i) \in \Pi(x)). \tag{2}$$

As this survey is preoccupied with telecommunication applications, it is of interest to define the network extension of the standard Nash equilibrium paradigm. To do so, consider first a strongly connected network, $G=(N,A)$, where N is the set of nodes of the network and A the set of links. Consider as well a set of users, or requests for connection, which are defined over node pairs, so that now $n \leqslant |N \times N|$, since in the simplest case, a single connection is established for each node pair. The strategy of a user, u^i, is then vector-valued, that is, $u^i = (u^i_1, \ldots, u^i_{m_i})$, for some m_i. Similarly, the vector of control parameters, x, is then a vector of vectors, each parameter type being defined over every node, link, or route of the network.

A natural variant of the Nash network equilibrium as defined above is one in which each node pair can accommodate several user classes, or differentiated traffic types. Clearly, in terms of the model, this is just a reformulation of the above with one more index to represent the user class or traffic type, or through a superposition of networks, one for each user type and coupled by constraints across user classes on the physical links. This multi-class or multi-user generalization does, however, have important consequences for the uniqueness of the equilibrium solution.

A final variant of the Nash equilibrium concept that we shall introduce here is that of multi-criteria equilibrium. In this setting, each user may have several criteria or utility functions to optimize. Let us denote the (now vector-valued) utility function of user i as $J^i = (J^i_1, \ldots, J^i_{p_i})$. We say that a vector y of dimension p *dominates* a vector z of the same dimension if, for any $j = 1, \ldots, p$ we have: $y_j \geqslant z_j$, with strict inequality holding for at least one j. In this case we write y dom z. Then, $\mathbf{u}^*$ is called a *multi-criteria*, or *Pareto–Nash*, *equilibrium*, if no user i can gain by unilaterally deviating (in the sense of the order "dom") from her or his strategy. In other words, for each i, there is no u^i such that

$$J^i(u^{-i*}, u^i, x) \text{ dom } J^i(\mathbf{u}^*, x).$$

Existence of Nash equilibrium is guaranteed under fairly mild conditions, if one allows for mixed, rather than pure or 0–1, strategies; for example, a Nash point can be shown to exist under the convexity and compactness of the strategy space and the semi-continuity of the utility functions together with some quasi-concavity properties, see e.g. [82].

2.2. Hierarchical, or Stackelberg, optimization

We now extend the framework to the case that a decision maker (who may represent, in telecommunication networks, the network administrator, the network designer, or a service provider) has an objective, possibly a vector-valued utility function, which she wishes to optimize. Among the components of this optimization objective there may be elements that coincide with the users' utilities, when the manager wishes to satisfy the users, and, for example, minimize their individual delays or loss probabilities. However, the manager is typically concerned not only with the efficient use of resources but also with purely economic considerations such as profit maximization.

The hierarchical relationship between the manager, on the one hand, who sets the parameters so as to achieve some objective, and the users who respond by seeking a new equilibrium, is modeled as a bilevel program, or a Stackelberg leader–follower problem [83]. Denote by $R(\mathbf{u}(x), x)$ the utility, or objective, of the manager. The function R depends on the parameters the manager sets, which we denote by x, and on the users' policy, strategy, or response to those parameters, $\mathbf{u}(x)$.

When the equilibrium $\mathbf{u}^*(x)$ defined in Section 2.1 exists and is unique, the objective of the network manager is to determine x that maximizes the function R, assuming that the users react to the parameters chosen, x, through their equilibrium actions $\mathbf{u}^*(x)$. In other words, the objective of the manager is to find

x^* that satisfies

$$R(\mathbf{u}^*(x^*), x^*) = \max_{x \in X} R(\mathbf{u}^*(x), x), \tag{3}$$

for some set of feasible actions, X. This problem class is tremendously useful, in principle, since it models the optimization that the decision maker wishes to perform simultaneously with the complex reactions of the users. However, it is also notoriously difficult to solve. When the users' equilibrium problem has constraints, even in its simplest form, the Stackelberg, or hierarchical, or bilevel, program, is fundamentally non-convex and non-differentiable. Showing existence of a solution to the hierarchical problem is also trickier than for the Nash equilibrium. See, for more details [84,85].

Several extensions and variations of the Stackelberg theme can be formulated as well. In the basic Stackelberg framework, the users and the manager have utility functions, J^i and R, respectively, that map from $\mathfrak{R}^n$ to $\mathfrak{R}$ [83]. However, in telecommunications applications, J^i and R may be vector-valued functions. Reinterpreting (3) for this Pareto–Nash framework case means that there does not exist a point x such that $R(\mathbf{u}^*(x), x)$ dom $R(\mathbf{u}^*(x^*), x^*)$.

Another extension arises when the equilibrium solution $\mathbf{u}^*(x) \in U^*(x)$ is not unique for every x. In this case, the problem (3) is not well-defined, since $R(\mathbf{u}(x), x)$ is no longer a function, but rather a point-to-set mapping. In that case, it is unclear to which value in $U^*(x)$ the decision maker should use in adjusting her or his control parameters, x. There are essentially two ways to reformulate the problem in this case so that it becomes well-defined [86]. In the first, the objective for the network may be to guarantee the best performance for any possible equilibrium, i.e. the decision maker is *pessimistic* (or assumes non-cooperative users) and therefore seeks an x^* that satisfies

$$R(\mathbf{u}^*(x^*), x^*) = \max_{x} \min_{\mathbf{u}^*(x) \in U^*(x)} R(\mathbf{u}^*(x), x). \tag{4}$$

On the other hand, if the decision maker is *optimistic* (or is in a cooperative setting), she may assume that the users will choose the equilibrium solution that favors her objective, in this case, maximization of R, giving the following problem: find x^* such that

$$R(\mathbf{u}^*(x^*), x^*) = \max_{x} \max_{\mathbf{u}^*(x) \in U^*(x)} R(\mathbf{u}^*(x), x). \tag{5}$$

Finally, one may consider the case of competition between several networks. This can give rise to a still more complex hierarchical game; taking into account the reactions $\mathbf{u}^*(x)$, of the n users to the decisions $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ of m network managers, the solution concept becomes an extension of Eq. (3) of the form

$$R^i(\mathbf{u}^*(\mathbf{x}^*), \mathbf{x}^*) = \max_{x^i} R^i(\mathbf{u}^*(\mathbf{x}^{-i*}, x^i), \mathbf{x}^{-i*}, x^i), \tag{6}$$

where $\mathbf{x}^{-i*} = (x^{1*}, \ldots, x^{i-1*}, x^{i+1*}, \ldots, x^{m*})$, R^i represents the utility (scalar or vector) of decision maker i, and x^i her or his decisions.

2.3. Potential games

In 1996, Monderer and Shapley [87] identified a class of games called "potential games". This class includes in particular several types of network routing games, such as the congestion games introduced in

[36] as well as the routing games in [1] used heavily throughout transportation planning (see in particular [88,89]). A game is a *potential game* if there exists a real-valued function on the decision space which measures exactly the difference in the utility that any user accrues if she or he is the only user to deviate. Mathematically, a potential game with n users is characterized by a potential function, $\Phi(\mathbf{u})$, such that for any user i, we have

$$J^i(u^i, u^{-i}) - J^i(v^i, u^{-i}) = \Phi(u^i, u^{-i}) - \Phi(v^i, u^{-i}).$$

The definition was extended in [88,89] to a finite number of classes, each of which has an infinite population of users. It is this latter setting that includes as a special case the equilibrium models in transportation, for which the Wardrop equilibrium, defined below, is the solution concept.

Potential games have nice properties, such as uniqueness of equilibrium and convergence of greedy algorithms to the equilibrium. This is discussed later in the context of networks in more detail.

2.4. Wardrop equilibrium

Network games have been studied in the context of road traffic since the 1950s, when Wardrop proposed his definition of a stable traffic flow on a transportation network [1]. The definition proposed by Wardrop was the following: "*The journey times on all the routes actually used are equal, and less than those which would be experienced by a single vehicle on any unused route*" (see p. 345 of [1]).

This definition of equilibrium is different than the one proposed by Nash. Expressing the Nash equilibrium in terms of network flows, one can say that *a network flow pattern is in Nash equilibrium if no individual decision maker on the network can change to a less costly strategy, or, route.* When the decision makers in a game are finite in number, a Nash equilibrium can be achieved without the costs of all used routes being equal, contrary to Wardrop's equilibrium principle. The Wardrop equilibrium assumes therefore that the contribution to costs or delays by any individual user is zero; in other words, the population of users is considered infinite. In some cases, Wardrop's principle represents a limiting case of the Nash equilibrium principle as the number of users becomes very large [90,58] (see also [88,91]). There are other ways to draw a parallel between the Wardrop and Nash equilibrium concepts, some of which define a "user" to be an origin-destination pair [58].

The Wardrop equilibrium falls into the category of potential games with an infinite number of users. Indeed, the Wardrop equilibrium condition can be expressed mathematically to state that the flow on every route r serving a commodity, or origin-destination (OD) pair, w, in the network is either zero, or its cost is equal to the minimum cost on that OD pair. The following system of equations is obtained from the following constraints (i) the cost on any route serving an OD pair is at least as high as the minimum cost on that OD pair (ii) a route serving an OD pair is not used if its cost is strictly larger than the minimum cost between that OD pair, and (iii) the demand for each OD pair is satisfied.

$$h_{wr}(c_{wr} - \pi_w) = 0, \quad r \in R_w, \quad w \in W, \tag{7}$$

$$c_{wr} - \pi_w \geqslant 0, \quad r \in R_w, \quad w \in W, \tag{8}$$

$$\sum_{r \in R_w} h_{wr} = d_w, \quad w \in W, \tag{9}$$

where h_{wr} is the flow on route $r \in R_w$, R_w is the set of routes joining node pair $w \in W$, and W is the set of origin-destination node–node pairs. The cost or delay on that route, r, is c_{wr}, and π_w is the minimum cost on any route joining node pair w. The demand for service between the node pair w is denoted d_w.

Then, adding non-negativity restrictions $h_{wr} \geqslant 0$ and $\pi_w \geqslant 0$, the resulting system of equalities and inequalities can be seen as the Karush–Kuhn–Tucker (KKT) optimality conditions of the following optimization problem, known as the Beckmann transformation:

$$\min \quad f(x) = \sum_{l \in A} \int_0^{x_l} t_l(x_l)\,\mathrm{d}x = \sum_{l \in A} \int_0^{\sum_{i \in I} x_{il}} t_l(x_l)\,\mathrm{d}x$$

subject to

$$\sum_{r \in R_w} h_{wr} = d_w, \quad w \in W, \tag{10}$$

$$\sum_{w \in W} \sum_{r \in R_w} h_{wr}\delta^l_{wr} = x_l, \quad l \in A, \tag{11}$$

$$x_l \geqslant 0, \quad l \in A, \tag{12}$$

where x_l is the flow on link l, x_{il} is the class-i flow on link l, I being the set of classes, and δ^l_{wr} is a 0–1 indicator function that takes the value 1 if and only if link l is present on route $r \in R_w$. In other words, contrary to the Nash equilibrium, the Wardrop equilibrium can be expressed as a single convex optimization program.

We may re-express the above classic definition of the Wardrop equilibrium in a way related to the definition of Nash equilibrium, i.e. as a minimization problem faced by each individual. All individuals belonging to population (travelers, packets or sessions) that have a given origin $s(i)$ and a given destination $d(i)$ face the same optimization problem. This population is called class i. The strategy set S^i of individuals in such a population is identified with all the paths in the network available between $s(i)$ and $d(i)$. The choice of a path is made by each one of the individuals. In the setting of Wardrop equilibrium, instead of describing the strategy of a given individual of a class (say class i), we define the amount of individuals within the class that use each strategy. We thus refer to the (class-i) strategy u^i as describing the behavior of all individuals in class i, so that u^i_j is the flow of individuals of class i that choose a path $j \in S^i$.

In the context of Wardrop equilibrium we refer typically to costs (delay) rather than utilities. Denote by $D_k(\mathbf{u})$, $k \in \{1, \ldots, m\}$, the delay (or cost) of path k. Then, letting $S^i_* \subset S^i$ be the subset of paths actually used by user i, i.e. the indices j such that $u^i_j > 0$, $\mathbf{u}^*$ is a Wardrop equilibrium if and only if it satisfies

$$\min_{k \in S^i} D_k(\mathbf{u}^*) = D_j(\mathbf{u}^*), \quad \forall j \in S^i_*, \ \forall i.$$

This type of model has been extended to a number of more general settings. In [92] and other references by its authors, the Wardrop equilibrium was extended to include link-level constraints, and in [19] to include different traffic classes, where delays in nodes or in links may depend on the traffic class.

Multiple user classes (in which the cost of using a link or a path depends on the user type) complicate the Wardrop equilibrium as well, since when cost functions depend upon more than one type of user flow, the set of KKT conditions above, one for each user class, need no longer correspond to the optimality conditions of a convex optimization problem (for special cases where a convex optimization is still applicable, see [93,94] and Theorem 3.4 in [35]). Rather, the multi-class KKT conditions can be stated compactly as a variational inequality (see Chapter 3 of [35]).

Another important variant of the Wardrop network equilibrium concept is the stochastic network equilibrium, which assumes that users make errors in their perceptions of delays and those errors are distributed according to some probability distribution around the true, mean delay (on each route). According to the probability distribution used, one obtains either the Gaussian (probit) equilibrium model or the Weibull (logit) model. (See, for example, [95] for a dated but still valuable introduction to the topic.) Unlike the basic and multi-class extensions, the stochastic network equilibrium concept does not appear to have been applied to date in communications applications.

On the importance of the concept of Wardrop equilibrium, we can learn from the numerous times that it has been reinvented. The results on Wardrop equilibrium were in fact obtained independently almost 50 years later in a context of mobile telecommunications in [96] and in the context of potential games in [88]. Wardrop-type principles were also obtained independently around thirty years before Wardrop in an economics, rather than network, context [97].

Nash equilibrium and Wardrop equilibrium are two extreme cases that can be modeled in networks. But also the combination of these may occur: some agents may have a large quantity of flow to ship (service providers that may control the routing decisions of all their users) while others agents (individual users who determine directly their routing) may have an infinitesimal amount of flow to ship. This scenario, along with the corresponding equilibrium notion, has been formalized and studied in [21,98,41,99].

Finally, we note that the hierarchical, or Stackelberg, or bilevel framework can encompass a Wardrop equilibrium governing the users' behavior in the same way as was defined in the Nash setting; the problem formulation (3) remains valid.

3. Non-cooperative service provisioning and network routing

In telecommunication networks, users can, in many cases, make decisions concerning routing, as well as the type and amount of resources that they wish to obtain. For example, in ATM architectures [100] used in high speed networks, the users decide on their type of service, be it CBR (constant bit rate), VBR (variable bit rate), or ABR (available bit rate). ABR, in contrast to CBR and VBR, is an *elastic* service, i.e. the user adapts her or his transmission rate to the state of the network; ABR is used, for example, in the present internet, through best-effort service.

In addition to choosing the type of service, the users may negotiate their quality of service (QoS), or performance parameters, namely, whether their quality guarantees are to be expressed in terms of PCR (peak cell rate), CLR (cell loss ratio), maximum delay, etc.

Different sets of parameters may suit the service requirements of a user. However, the performance measures (such as throughput, CLR, delay) depend not only on the user's choices in establishing the communication, but also on the decisions of other connected users, where this dependence is often described as a function of some network "state". For example, the available resources and the delay of a best-effort type connection, such as ABR, depend not only on the user's own choices, but clearly also upon the choices made by other users. In this setting, the game paradigm becomes a natural choice, at the user level.

Constrained Nash equilibrium is quite natural in the context of, for example, ATM architectures, where users express their requirements for quality of service by bounds they wish to have on delays, CLR, etc. For interactive audio applications, for example, the quality of the communication is insensitive to delay, as long as it is below approximately 100 msec. An audio application could therefore seek to (selfishly) minimize losses, subject to a maximum bound on the delay it experiences. Such constrained Nash equilibria have been studied in telecommunications and internet provisioning applications (e.g. [51,101–104,14]).

Next, we present a basic structure that many network games have in common, along with several examples.

3.1. Framework of a service provisioning game

Many games arising in networks may be modeled as follows. There are n applications or users, and m service classes. Application, or user, i has a traffic of rate Λ^i, and has to determine how to split it between a subset S^i of service classes available to that user (application). A strategy of user i is given by an allocation vector $u^i = (u^i_1, u^i_2, \ldots, u^i_m)$ where u^i_j is the amount of traffic that user i assigns to service class j. The set of policies for application i is given by the simplex $\{u^i \in \Re^m | \sum_{j=1}^m u^i_j = \Lambda^i, u^i_j \geqslant 0, \;\; j = 1, \ldots, m\}$.

This framework has been used in particular in the contexts of service provisioning [105] and routing games [58,34].

In [105], there was no explicit use of the network. In that reference, $S^i = \{1, \ldots, m\}$ for all users, and the utility function for using any service class is binary valued and are defined as follows:

- There is a QoS (Quality of Service) q_j defined for each *service class*, j, which is a monotone function of the summation over all users (applications) of that service class: $\sum_{i=1}^n u^i_j$.
- The utility for user i of assigning u^i_j to class j is given by $J_i(u^i_j, q_j)$, which is assumed to be monotone in both arguments.[1] The global utility for class i is the sum over j of $J_i(u^i_j, q_j)$.

This corresponds to a distinction between acceptable versus unacceptable QoS. The goal of a user in [105] was to maximize the fraction of the traffic that receives acceptable QoS. This gives rise to non-concave utilities and hence to cases of nonexistence of an equilibrium. Sufficient conditions are given in that reference for the existence of equilibria, and an extension to multidimensional QoS for each user was presented. Moreover, there are some results on the convergence of greedy update policies to the equilibrium.

3.2. Routing games

A problem somewhat related to [105], yet with significantly more complex utility functions, occurs when the network itself is incorporated into the model. In this case, each user has a given amount of flow to ship and has several paths through which he may split that flow. Such a routing game may be handled by models similar to [105] in the special case of a topology of parallel links. This type of topology is studied in detail in the first part of [34] as well as in [19]. However, the model of [105] does not extend directly to other topologies. Indeed, in more general topologies, the delay over a *path* depends on how much traffic is sent by other users on any other path that shares common links.

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[1] More precisely, $J_i(u^i_j, q_j)$ is monotone increasing in its first argument, and if q_j represents a “desirable” feature then $J_i(u^i_j, q_j)$ is monotone increasing in q_j as well; also in that case q_j is monotone decreasing in $\sum_{i=1}^n u^i_j$. Note however that in [105], q_j stood for loss probabilities, which stands for a “negative” feature, so in fact $J_i(u^i_j, q_j)$ was taken to be monotone decreasing in q_j and q_j was monotone increasing in $\sum_{i=1}^n u^i_j$.

Routing games with general topologies have been studied, for example, in [58], in the second part of [34], as well as in [19]. A related model was studied thirty years ago in [36,37] in a discrete setting. Rosenthal proposed a discrete approach to the network equilibrium model; in his setting, there are n players, where each has one unit to ship from an origin to a destination and wants to minimize her transport cost (which is the sum of the link costs used). It is shown that in such a model there always exists a pure strategy Nash equilibrium. He introduces a kind of discrete potential function for computing the equilibrium. Nevertheless, if a player has more than one unit to ship, such an equilibrium does not always exist.

The paper [49] considers a multi-user network shared by non-cooperative users, in which each user reserves some resource in order to establish a virtual path. Users are non-cooperative: each user seeks to optimize her or his own selfish utility, which includes the guaranteed quality of service, as well as the cost incurred for reserving the resource. For the case of a shared resource (the total resource available to users modeled by a single link), existence and uniqueness of the Nash equilibrium is proved. The authors establish the convergence to this unique equilibrium under Gauss-Seidel and Jacobi schemes. For a general network, users may be sharing more than one resource and each user would have preferences among several links; the authors extend the results of the one-resource model to various general network topologies. The formal results are tested by simulating the schemes on an experimental network.

In the transportation sector, this is the classic *fixed demand* equilibrium routing model, described above and formulated initially by [1]. See [35] for an extensive list of references using this paradigm in the transportation literature.

4. Non cooperative flow control games

Flow control problems have been considered in different settings, both in dynamic as well as static contexts. By “dynamic” we mean that the decisions of users depend on some observed state of the network, which may vary dynamically. Flow control can often appear as part of a routing game where both routes as well as quantity (or rates) to be shipped should be determined.

4.1. Static flow control

The static flow control problem is related to the question of what should be the average transmission rate of a user. It is known that this type of problem can often be handled as part of routing problems in which one wishes to determine how much traffic should be sent over each path in the network; if we do not impose a demand constraint (stating that the sum of flows sent over all paths should be a given constant) then the solution to this routing problem clearly provides at the same time the solution of the flow control problem. Thus routing and flow control decisions can be done simultaneously, and in the same framework as discussed before, i.e. of routing games.

Indeed, in the context of transportation equilibrium models, the demand level of users between node pairs is given by a function that depends upon the state of the network, which in turn depends upon the routing decisions. In this manner, the amount of flow to route on the network becomes a variable whose value is set optimally, simultaneously with the routes, as a function of the network characteristics and the demand function. This is referred to as the *elastic demand* equilibrium model; for references, see [58,35] and references therein. Another example of that approach can be found in [59].

An important feature in all of the above references is that costs are given in terms of the sum of link costs; that is, route costs are additive functions of the constituent links' costs. This assumption simplifies considerably the resolution of the routing and flow control-routing models by allowing the use of highly efficient shortest-path algorithms to solve the subproblems. Indeed, when interactions across users or applications are held fixed, the resulting routing and flow control-routing problems can be expressed as pure shortest path problems.

There are models, however, in which this additivity of the route costs is not an acceptable assumption. For example, in cases where the basic building blocks of the overall utilities are not link delays (or link costs), but instead the ratio between overall throughput (or some power of it) to overall delay. This is the well known *power criterion*, and it has been frequently used in flow control problems and in games, see [182–186]. In the non-cooperative context, some variant of the power criterion has been used in [60], in which the utility is related to the sum of powers over the links. The part of the utility in [60] that corresponds to the delay is given by the sum of all link capacities minus all link flows, multiplied by some entropy function. While this utility does not directly reflect the actual expected delay, it has the advantage of giving rise to computable Nash equilibria in the case of parallel links for the combined flow-routing game.

In [56] the actual power criterion is considered, i.e. the ratio between (some increasing function of) the total throughput of a user and the average delay experienced by traffic of that user. The equilibrium for the flow-routing game is obtained for the limiting case as the number of users becomes very large. The limit is obtained explicitly; there are cases, however, where two equilibria are obtained.

4.2. Dynamic models

Several non-cooperative flow control models have been proposed and analyzed in a dynamic context. Important references are [13,14], which consider a network with a general topology where each source has a window end-to-end flow control. The available information for a user is thus the number of packets within the network not yet acknowledged. Each user wishes to maximize the throughput for her or his own flow, and also would like her delay to be bounded by some given value. Thus each user faces a constrained optimization problem. The equilibrium obtained is decentralized since each user has only local information on her own unacknowledged packets. Hsiao and Lazar [13] obtained threshold equilibrium policies for this problem using the product form of the network as well as the Northon's equivalence approach that allows one to reduce a network to an equivalent single queue. The threshold policy is then obtained through coupled linear programming problems. The existence of an equilibrium is established in [14]. A more general theoretical framework for equilibria with constraints in stochastic games is proposed in [106].

In [5], rate-based flow control is considered in which each user can dynamically vary her or his transmission rate. The available information is assumed to be the queue length (or equivalently, the delay) at the bottleneck queue. The total available bandwidth to all controlled sources at this node is assumed to be the node capacity minus the bandwidth used by higher priority traffic. Typical performance measures are throughput, to be maximized, and overflow, to be minimized. Note that we may lose in throughput if the queue is empty, and lose packets if it is full. A good trade-off between these can be obtained by setting an appropriate target queue length and trying to track it.

Another possible performance measure may be related to how well the input rate of a connection tracks its share of the available bandwidth. By considering an immediate cost per user, the problem is cast into the framework of linear quadratic dynamic games. One such cost is obtained by taking a weighted sum of two objectives: the square of the difference between the queue size and its target value, and the square of the difference between the input rate of a connection and its available bandwidth. In [5], an equilibrium policy is shown to exist and to be unique; moreover it is explicitly computed along with the resulting performance measures.

Another type of dynamic flow control (combined with routing) is considered in [99]. The players have to ship a given amount of flow within a certain period, and can decide dynamically at what rate to ship at each instant. A dynamic mixed equilibrium is computed, where *mixed* refers to the combination of both infinitesimal, as in the Wardrop paradigm, and “large” users, the latter being modeled through the Nash setting. In the transportation context, many other dynamic routing models have been developed, most using the Wardrop equilibrium context. One textbook on the subject is [107].

5. Uniqueness of the equilibrium

The two first questions that arise in networking games are those of the existence and the uniqueness of equilibria. We focus in this section on the uniqueness problem as the existence is usually much easier to establish using standard fixed point theorems. For example, in [34] the existence of equilibria in routing games is established for general cost functions and general topology, whereas its uniqueness is obtained for very special cases.

The uniqueness of an equilibrium is quite a desirable property, if we wish to predict what will be the network behavior. This is particularly important in the context of network administration and management, where we are interested in optimally setting the network design parameters, taking into account their impact on the performance in equilibrium.

For routing games in networks, in the context of the Wardrop assumption of an infinite population of users, the uniqueness of the equilibrium [1] has long been known in some weak sense. Indeed, since the model can, in its simplest setting, be cast as a single convex optimization problem, optimization theory tells that when the objective is strictly convex and the feasible region convex, the solution exists and is unique. Even when the underlying Wardrop equilibrium model is more complex, for example, modeling multiple user classes, so that the equivalent convex optimization transformation no longer applies, variational inequality theory still tells us that the solution is unique when the cost mapping is globally strongly monotone. Unfortunately, that latter assumption is rarely satisfied for general multi-class problems. Indeed, it is no longer sufficient in the multi-class case for each class’ delay function to be increasing (or each users’ utility to be decreasing); rather it is necessary for the *overall* delay vector or utility vector to be strongly monotone (a formal definition will be given in Eq. (13)) which is a much stronger assumption, and one related to the diagonal dominance of the Jacobian matrix of the delay or utility function mapping.

The uniqueness of Wardrop equilibrium holds in a weak sense: it is the *total link utilization* that is unique, rather than the flow of each user on each link. Only in special cases is the flow on each path also uniquely determined, such as is the case for the stochastic Wardrop equilibrium model, see first paragraph of p. 64 in [35]. Uniqueness of the Wardrop equilibrium was shown to hold for particular multi-class networks (i.e. networks in which there are several classes of users and the delay in a node or a link may depend on the class) in [19,27].

As shown in [88,89], the setting of Wardrop turns out to be a potential game. The uniqueness of equilibrium in potential games was established in [108]; further, the equilibrium is shown to be unique not only for Nash equilibria but also in the larger class of correlated equilibria. Note however, that in [108] only models with finitely many users were considered, and thus it does not directly cover the framework of Wardrop.

Uniqueness of Wardrop-type equilibrium has been obtained in some other related problems. Cominetti and Correa [109] considered a transportation network with an origin, a destination and n bus lines between them. They analyze this model with an infinite population of users and hence are interested in the Wardrop equilibrium. In their model, a bus line is characterized by two parameters, its in-vehicle travel time and its frequency. Passengers choose not a single route, but rather a set of lines, and board the first available bus in that set. Due to congestion, the decision of each passenger depends upon the decisions of the other passengers. Under general assumptions, the authors obtain the existence and uniqueness of the equilibrium.

As mentioned above, realistic models for which we have uniqueness of the equilibrium are quite unusual. In fact, a simple counter-example of a network with four nodes is given in [34], and a two-node two-class Wardrop network example is discussed in [110]. It is thus not surprising that much effort has been given to understand the conditions under which there is uniqueness of the equilibrium.

A quite powerful tool for establishing uniqueness is the framework of [81] who introduced the concept of DSC (diagonal strict concavity); this is a weak version of concavity which is defined for a multi-user setting each with its own utility. DSC states that the weighted utility function gradient, given by the vector whose elements are $g_i = \zeta_i \partial J^i(\mathbf{u}, x)/\partial u_i$, for some vector $\zeta > 0$, satisfies

$$(\hat{\mathbf{u}} - \bar{\mathbf{u}})^T [g(\hat{\mathbf{u}}, \zeta) - g(\bar{\mathbf{u}}, \zeta)] > 0, \tag{13}$$

which is the strict monotonicity of the scaled mapping g. Note that if J^i did not depend on i then (13) would imply the standard notion of concavity of J^i. The diagonal dominance of the Jacobian, or matrix of partial gradients, of g is a sufficient condition for the strict monotonicity of g. As we mentioned previously, this condition typically does not hold in routing games. However, there are a few cases in which it has been shown to hold: (i) the problem of two users routing into two parallel queues for which the DSC conditions are shown in [34] to hold in the case of light traffic, and (ii) a network with general topology with certain polynomial costs [17].

In the absence of other general tools for establishing uniqueness, and in view of counterexamples that show that there are cases in which it fails, the study of uniqueness has become a complex case-by-case study. For some topologies, uniqueness has been obtained for quite general cost functions; notably, for the case of parallel links [34] and for topologies arising from distributed computing with communication lines, see [39] and references therein. Uniqueness has also been established for symmetric users [34].

Another interesting result related to uniqueness is the following. Assume that there are two equilibria with each having the following property: a user sends positive flow over some link if and only if all other users also send positive flow over that link. Then the two equilibria coincide. This has been established in [34] and further extended in [19].

Although the study of equilibria is more involved in the case of a finite number of users than in the infinite, Wardrop, setting, the uniqueness results obtained (in all of the above references) for the finite case are *stronger* than for the infinite case. In particular, the uniqueness is in the sense of the amount of flow that is sent by each user through each path, rather than in terms of the total link utilization.

Finally, some recent uniqueness results have been established in [21], for a general topology, and in [41], for some particular topology, for the mixed equilibrium case, that is the setting of both Nash and Wardrop equilibrium paradigms jointly coexisting on a network.

6. Convergence to the equilibrium

The equilibrium has a meaning in practice only if one can assume that it is actually reached from non-equilibria states, since there is no reason to expect a system to be initially at equilibrium. Several approaches have been proposed in the literature to obtain convergence. Some rely on update policies that have centralized characteristics (in terms of synchronization between the order of update); an example is the round robin update order. Other approaches establish convergence under asynchronous best response mechanisms. It appears that the latter are more appropriate for describing a real decentralized non-cooperative system.

Rosen [81], who considered the case of a finite number of players, established the convergence of a dynamic scheme in which the policies are updated continuously (in time) by all users so as to move in the direction of the gradient of the performance measure. In the case of a unique Nash equilibrium, this scheme is shown to converge to that equilibrium. In the case of multiple equilibria, this procedure converges to one of the equilibria, and it is possible to predict to which equilibrium it will converge. As already mentioned, the conditions under which Rosen's setting holds in networks are quite restrictive.

An alternative approach for the dynamic convergence of greedy policies to an equilibrium (even in the absence of a unique equilibrium) is in the class of submodular games and supermodular games [111,112]. In [22], the authors study a load balancing problem where it is shown that, depending on the parameters, the costs are either submodular or supermodular. In both cases greedy algorithms are shown to converge to the unique equilibrium. Examples of convergence in both a submodular setting as well as in supermodular games (and their combination) in simple queuing problems are presented in [112].

We note that, in the field of transportation equilibrium, supermodularity is not the concept used for proving uniqueness or convergence. Rather, monotonicity and its variants are the preferred concepts. While the two notions are related, it may be possible to develop stronger results by making use of one or the other, in particular through the use of some weaker forms of monotonicity such as pseudo-monotonicity or nested monotonicity [113,110]. See [35] for a comprehensive description of the basic definitions and [114] for a more advanced compendium of the role of and forms of monotonicity.

In [115], Shenker considers a non-cooperative model with a single server (exponential) and several sources (the users, who are Poisson). The utility of a user is a function of the amount of service received and the queue length (i.e. congestion). The author concludes that no service discipline can guarantee optimal efficiency, and that a service discipline called *Fair Share* guarantees fairness, uniqueness of Nash equilibrium and robust convergence.

For routing games with an infinite population of players, it has been shown that greedy updates converge for quite general costs and for general topology; this was shown in fact for the larger class of potential games [88,89].

In [18] a very simple case of convergence is considered: that of n users routing to two parallel links. The link costs considered are linear. Both random (asynchronous) greedy as well as round-robin policies are shown to converge to the equilibrium. However, it is also shown that if more than three players update simultaneously their routing strategies, then this results in diverging oscillations. To avoid such oscillations in the case of simultaneous updates, one has to use relaxation, or smoothing, i.e. each user should apply at each update some linear combination between the previous strategy and the best response one.

Greedy updates have been shown in [22] to converge in a simple setting of distributed computing: a network represented by three nodes and three links (two sources of arrival of tasks, and one destination node; the links between sources and destination represent computers, whereas the links between the sources represent a communication line).

We also mention here the paper [116], that considers the problem of how to split a file between several computers; the decisions are taken in a distributed way by the computers themselves (this involves processing and communication delays). Although there is one global objective that is optimized, this problem has some interesting features of a game (or a team) problem since decisions are distributed. The algorithms compared belong to the class of resource-directive approaches, where at each iteration the marginal value of the resource is computed using the current allocation, by each computer in parallel, then an exchange of this computed value is made between all the computers.

7. Braess paradox, pricing, and Stackelberg Equilibrium

7.1. The Braess paradox

The service providers or the network administrator may often be faced with decisions related to upgrading of the network. For example, where should one add capacity? Where should one add new links?

A frequently used heuristic approach for upgrading a network is through *bottleneck analysis*, where a system bottleneck is defined as "a resource or service facility whose capacity seriously limits the performance of the entire system" (see p. 13 of [117]). Bottleneck analysis consists of adding capacity to identified bottlenecks until they cease to be bottlenecks. In a non-cooperative framework, however, this heuristic approach may have devastating effects; adding capacity to a link (and in particular, to a bottleneck link) may cause delays of all users to increase; in an economic context in which users pay the service provider, this may further cause a decrease in the revenues of the provider. This problem was identified by Braess [118] in the transportation context, and has become known as the *Braess paradox*. See also [119,120]. The Braess paradox has been studied as well in the context of queuing networks [20,23–25,121].

In the latter references both queuing delay as well as rejection probabilities were considered as performance measures. The impact of the Braess paradox on the bottleneck link in a queuing context as well as the paradoxical impact on the service provider have been studied in [59]. In all the above references, the paradoxical behavior occurs in models in which the number of users is infinitely large and the equilibrium concept is that of Wardrop equilibrium, see [1].

It has been shown, however, in [29,32], that the problem may occur also in models involving a finite number of players (e.g. service providers) for which the Nash framework is used. The Braess paradox has further been identified and studied in the context of distributed computing [39,40,122] where arrivals of jobs may be routed and performed on different processors. Interestingly, in those applications, the paradox often does not occur in the context of Wardrop equilibria; see [39].

In [123] (see also [38]), it was shown that the decrease in performance due to the Braess paradox can be arbitrarily larger than the best possible network performance, but the authors showed also that the performance decrease is no more than that which occurs if twice as much traffic is routed. The result was extended and elaborated upon in more recent papers by the same authors. In [124], a comment on the results of [123] was made in which it is shown that if TCP or other congestion control is used, rather than agents choosing their own transmission rates, then the Braess phenomenon is reduced considerably. Indeed, this conclusion can be reached intuitively by considering (as is well known in the study of transportation equilibria) that the system optimal equilibrium model (in which the sum of all delays are minimized) does not exhibit the Braess paradox; congestion control serves to force transmission rates to such a system optimal operating point.

An updated list of references on the Braess paradox is kept in Braess' home page at http://homepage.ruhr-uni-bochum.de/Dietrich.Braess/#paradox.

7.2. Architecting equilibria and network upgrade

The Braess paradox illustrates that the network designer, the service provider, or, more generally, whoever is responsible for setting the network topology and link capacities, should take into consideration the reaction of (non-cooperative) users to her or his decisions. Some guidelines for upgrading networks in light of this have been proposed in [125,126,29,31,32], so as to avoid the Braess paradox, or so as to obtain a better performance. Another approach to dealing with the Braess paradox is to answer the question of which link in a network should be upgraded; see, for example, [59] who computes the gradient of the performance with respect to link capacities.

A more ambitious aim is to drive the equilibrium to a socially optimal solution. In [29] this is carried out under the assumption that a central manager of the network has some small amount of his or her own flow to be shipped in the network. It is then shown that the manager's routing decision concerning his own flow can be taken in a way so that the equilibrium corresponding to the remaining flows attain a socially optimal solution.

7.3. Pricing

An alternative approach to obtaining efficient operating solutions is through pricing.

A naive approach for pricing could be to compute an optimal policy for the network as a whole and simply impose a high fine on any user that deviates from it. This approach would require, however, centralized computation and signaling that would be difficult to implement. Therefore, research on pricing schemes in recent years has focused on methods to charge locally (at each link or node) for the resources used, under the assumption that such local data are easy to measure and impose.

It is well known, in the setting of Wardrop equilibria, that adding a fee equivalent to the *marginal cost* of the delay function to the user delay on each link renders the solution of the Wardrop equilibrium problem equal to that of the system optimal problem. A similar approach was taken in [127,128] in telecommunications, using the context of Wardrop-type equilibria. Similarly, it was shown in [129], in the context of a finite number of users, that if the price at each link is chosen to be proportional to the congestion level at the link, then efficient equilibria are obtained.

The next few references seek a vector of prices that achieves an objective similar to that of the system optimal solution, described above for Wardrop equilibria.

Orda and Shimkin [130] study the case of many selfish users, each one wishing to ship her traffic through some service class. It is then assumed that the intent of the service provider is to have a unique allocation of each traffic type to one of the service classes; such an allocation is called the *nominal flow allocation*. Pricing is used to induce users to choose the service class which is adapted to their needs (QoS) and moreover which corresponds to the intent of the network service provider. Orda and Shimkin establish a necessary and sufficient condition for the existence of prices such that the user-optimal flow allocation is unique and coincides with the nominal flow allocation.

Low and Lapsley [131] consider a model where S sources share a network. Each source (i.e. user) s has a path and a utility function; the source s chooses its transmission rate in order to maximize its own utility. The goal is to propose a set of prices that induces the maximization of the global utility. Again, this is similar in spirit to the idea of *marginal cost pricing* discussed above with respect to Wardrop equilibrium and also similar to the model of [132]. In [133], it is shown that the link prices of schemes such as those proposed in [132,131] are in general not unique in networks. The theoretical justification for this result and an example are provided in which particular prices may be easily obtained, and those prices appropriately defined can be unique. In [134], certain choices of pricing objectives in this context, such as revenue-maximizing prices are presented and analyzed.

Low [135] considers a single node with an allocation scheme that provides each user with a fixed minimum and a random extra amount of bandwidth and buffer capacity; the network then sets prices on the resources. Two models are proposed: in the first one, each user has an initial allocation and seeks a new allocation maximizing his own utility under the constraint that the new allocation's price is the same as the initial allocation's price; in the second one, the above constraint is absent. It is shown, for the first model, that at equilibrium all users have positive variable allocation in bandwidth and buffer capacity. For the second model, some properties of the equilibrium are exhibited.

A related problem is studied by Chen and Park [136]. They assume that a routing is given and the network provides service classes at each switch; with each service class is associated a price. Users have to choose a service class in order to satisfy (at the lowest price) their QoS requirement. In this context, the authors propose an architecture for non-cooperative multi-class QoS provision.

Pricing has also been used as a tool for obtaining efficient equilibria when demand is controlled, rather than in pure routing, in [137–139]. With the rapid growth of the Internet and its evolution from a heavily subsidized network to a commercial enterprise, much attention has been given to pricing the demand, see for example, [140–145,104].

Pricing schemes for attaining efficient equilibria, where both demand and routing are controlled, have been considered in [59,146].

Pricing is used in another context in [143], where the authors model an ATM network using a microeconomic paradigm. The network offers bandwidth and buffers for rent. The users have to ask, and pay, for the amount of these resources that can provide them the QoS they require. The authors assume that each user knows a bound on the burstiness of her or his connection and also knows the minimum bandwidth μ required for the connection. The authors propose an algorithm that converges to a unique, optimal allocation and service provisioning procedure that prevents cell loss.

Some other references on pricing in networks are [147–164].

7.4. Hierarchical, or Stackelberg, equilibrium in telecommunications

One further step in the interaction between the manager (who represents the network designer or operator) and the users, is to assume that the former is interested not just in attaining an efficient equilibria for the latter, but may have her own objectives (such as maximizing revenue).

In the telecommunication context, this framework has been studied in [165,30,33,59,50,183–185].

When the equilibrium problem involves a constrained routing, or control-routing, problem, the user level solution of the hierarchical, or Stackelberg, equilibrium problem cannot be expressed analytically in closed form. In that case, the optimization of the network manager's problem is implicit and further nonconvex; in other words, it does not posses a unique optimum, and its algorithmic solution is quite time consuming.

A different approach was proposed in [166] for transportation networks and studied within the context of internet-type networks in [134]. The idea is to solve a resource allocation, or routing, problem in which link capacity constraints are Lagrangian relaxed, for a unique optimal solution. The uniqueness of the optimal routing holds under conditions discussed above. Then, taking prices to be the Lagrange multiplier values, those prices are optimized from the point of view of the network manager. This pair of coupled problems has a unique solution when the equilibrium routing problem does, and can be computed in time proportional to solving the original routing problem.

8. Cooperative games and resource sharing

Questions of how to share common resources, or how to share the cost of constructing a network, typically fall into the realm of cooperative games; see e.g. [167–172].

In [46,50] the problem of bandwidth sharing between different users is considered. A general network topology is studied, and the question is how much bandwidth, or extra capacity, should be allocated by the network to each user at each link. These papers propose the Nash Bargaining concept [173,174] for assigning this capacity. This concept is characterized by the following properties: (1) it is Pareto-optimal, (2) it is scale invariant, i.e. the bargaining solution is unchanged if the performance objectives are linearly scaled, (3) the solution is not affected by enlarging the domain if agreement can be found on a restricted domain, and (4) the bargaining point is symmetric, i.e. does not depend on the specific labels: users with the same lower bounds and objectives receive the same share. It is shown that this sharing of the bandwidth has the proportional fairness property introduced in [175], and is unique. Pricing was also considered in [50]; the proposed scheme is such that a user is never charged more than her or his declared budget but could be charged less if the amount of congestion in the network links used by the connection is low.

The idea of using the Nash bargaining solution in the context of telecommunication networks was first presented in the context of flow control in [184]. The Nash bargaining concept has been recently used in [47] for pricing purposes, where the solution concept is used to identify a pricing strategy in which the two players are the service provider and the set of all users. In [47], only simple network topologies are considered. However, the analysis in [47] considers also the case of several user priorities which models the possibility for the service provider to offer different qualities of services at different prices. Another application of the Nash bargaining concept in networking can be found in [176].

The third property of the Nash bargaining solution has received criticism since it implies that a player does not care how much other players have given up. (This is related to the fact that the Nash bargaining concept takes into account required lower bounds but not how far the solution is from any upper bound.) Two alternative notions of fair sharing have thus been introduced with properties 1, 2 and 4 of the Nash bargaining solution, but with a variation of the third property, namely, the modified Thomson solution and the Raffia–Kalai–Smorodinsky solution. A unified treatment of the Nash solution as well as of these two has been introduced in [177] for two players and extended in [178] for the multi-person case. These concepts have been applied to Internet pricing in [179].

Another concept in cooperative games for sharing resources is the Aumann–Shapley pricing, which has desirable properties such as Pareto optimality. Haviv [48] proposes this approach to allocating congestion costs in a single node under various queuing disciplines.

Finally, we cite recent work in non-cooperative resource allocation which uses marginal cost and Shapley values without the assumption that choices will be in equilibrium. Instead, it is assumed that sometimes equilibrium will not be reachable, so authors have looked into ensuring that users always choose efficient allocations by making those choices dominant irrespective of other users' choices. This has been referred to as *strategy proof* or *incentive-compatible* mechanisms. See, for example, [180,181].

9. Synthesis and conclusions

As stated in the introduction, numerous results have been invented and reinvented in different communities, under different names, and with varying degrees of generality. This survey attempts to provide some synthesis across communities of some of these results. Certainly, more synthesis and *unification* would be a positive stimulus to this branch of science.

Examples of similar models and results across communities include, among others, the areas of potential and congestion games in game theory and the traffic equilibrium model of transportation science. While the former field has made great strides in generalizing this form of a game, the form of the potential, and developing the sophisticated notion of supermodularity to study it, the latter field has generalized rather in a different sense, eliminating the potential and tending toward variational inequalities, and hence the notion of monotonicity (and its variants) for its analysis. It seems desirable to merge some of these complementary developments and apply them as well to the communications arena.

In terms of stochastics, telecommunication applications and game theory have included random variables in their models in quite a different way from applications in transportation. In the former cases, random arrival rates or usage levels are modeled through exponential or other distributions and expected values are generally used or derived in such a way that often limits the size of the networks that can be handled. In the transportation literature, stochastic models based on the logit (Weibull), in particular, and also probit (Gaussian) distributions have been extended to the network setting and exact and approximate algorithms devised, even for large-scale networks. This appears to be a promising avenue for future development in the telecommunications arena.

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